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Two-component eigenfunction expansion for open systems described by the wave equation II: linear space structure

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Abstract. For a broad class of open systems described by the wave equation, the eigenfunctions (which are quasinormal modes) provide a complete basis for simultaneously expanding outgoing wavefunctions ϕ. In this paper, the linear space structure associated with this expansion is developed. Under a modified inner product, the time-evolution operator is self-adjoint, even though energy is not conserved for the system alone. Thus, the eigenfunctions are mutually orthogonal. Consequently, the usual tools of eigenfunction expansions can be transcribed to these open systems. As an example, the time-independent perturbation theory is developed in straightforward analogy with quantum mechanics, giving the shift in both the real part and the imaginary part of the eigenvalues ω.

1. Introduction

In the preceding paper [1], hereafter referred to as paper I, the concept of eigenfunction expansion has been developed for *open* systems described by the wave equation $[\rho(x)\partial_t^2 - \partial_x^2]\phi(x,t) = 0$. The wave equation is considered on a finite interval $I = [0, a]$, representing, for instance, an optical cavity. Attention is restricted to systems that have a discontinuity at $x = a$ (say a step), and have 'no tail' beyond $x = a$, i.e. $\rho(x) = 1$ for $x > a$. The boundary conditions for the wave equation are (a) the wavefunction vanishes at $x = 0$, and (b) the wavefunction satisfies the outgoing wavecondition at $x = a^+$. (In paper I, the generalization to waves that are outgoing at both ends of the interval was briefly sketched, and the results of the present paper can likewise be generalized, in a straightforward manner that will not be spelt out.)

For such a finite interval, the spectrum becomes discrete; heuristically, the eigenfrequencies, ω, are spaced by $\Delta\omega \sim \pi/a$. Computationally and conceptually, this is much more convenient than dealing with an infinite (or semi-infinite) line, for which the spectrum would be continuous in ω. The consequence is that for the finite interval, energy is no longer conserved, and mathematically the time-evolution operator is no longer Hermitian in the usual sense. Thus, the usual framework of eigenfunction expansions for Hermitian systems is not immediately applicable. Moreover, the eigenfunctions are now quasinormal modes (QNMs) with complex frequencies, rather than normal modes (NMs). Nevertheless, it was shown in paper I that for systems with discontinuities but 'no tail', the eigenfunction expansion is valid and complete. However, the uniqueness of that expansion was not dealt with in paper I.

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The central theme of paper I was that one has to consider the simultaneous expansion of a pair of functions:

$$\begin{pmatrix} \phi(x) \\ \hat{\phi}(x) \end{pmatrix} = \sum_n a_n \begin{pmatrix} 1 \\ -\mathrm{i}\omega_n \rho(x) \end{pmatrix} f_n(x) \tag{1.1}$$

where $\hat{\phi}$ should be thought of as the conjugate momentum $\rho(x)\partial_t \phi(x,t)$. The eigenfunctions, f_n, and the corresponding eigenvalues, ω_n, satisfy

$$\left[\frac{\partial^2}{\partial x^2} + \rho(x)\omega_n^2\right] f_n(x) = 0 \tag{1.2}$$

while the corresponding second components are

$$\hat{f}_n(x) = -\mathrm{i}\omega_n \rho(x) f_n(x). \tag{1.3}$$

The second component is needed because in order to solve the dynamics one must treat the initial $\partial_t \phi$ on the same footing as the initial ϕ. Moreover, the outgoing wavecondition cannot be specified in terms of a single function; in fact, this condition is stated as the following relation between ϕ and $\hat{\phi}$: $\hat{\phi}(x = a^+) = -\phi'(x = a^+)$.

The main result of paper I was the completeness of the eigenfunction expansion (1.1), with the expansion coefficients given by the projection

$$a_n = \frac{\mathrm{i}}{2\omega_n}\left\{\int_0^{a^+} \mathrm{d}y\,[\hat{f}_n(y)\phi(y) + f_n(y)\hat{\phi}(y)] + f_n(a)\phi(a)\right\}. \tag{1.4}$$

The normalization convention is $\langle f_n | f_n \rangle = 2\omega_n$, where the generalized norm is defined by

$$\langle f_n | f_n \rangle = 2\omega_n \int_0^R \mathrm{d}x\, \rho(x) f_n(x)^2 + \mathrm{i} f_n(R)^2 \tag{1.5}$$

for any $R > a$. This generalized norm refers explicitly to ω_n, and is therefore not immediately applicable to wavefunctions which are not eigenfunctions, nor immediately generalizable to an inner product.

The present paper develops further, in section 2, the linear space structure that supports these concepts. Central to these developments is the introduction of a generalized inner product. This has all the usual properties, except that it is linear, rather than conjugate linear, in the bra vector. In section 2 we shall show that this inner product has the following desirable properties. (a) It agrees with the generalized norm (1.5) for the inner product of an eigenfunction with itself. (b) The projection of $(\phi, \hat{\phi})$ in (1.4) is precisely the inner product with the eigenfunctions. (c) Time evolution can be written as a first-order equation involving a Hamiltonian operator, $\mathcal{H}$, formally analogous to the Schrödinger equation. Most importantly, $\mathcal{H}$ is self-adjoint under this inner product, even though the system is nonconservative. (d) Consequently the eigenfunctions are mutually orthogonal, and as an important corollary, the eigenfunction expansion in paper I is unique.

As a result of these properties, many of the usual tools of mathematical physics for eigenfunction expansions can be transcribed, even though energy is not conserved. As an example, section 3 presents the time-independent perturbation theory in terms of the discrete set of eigenfunctions, giving the shift in both the real part as well as in the imaginary part of the eigenvalues ω_n. An example of the perturbative formalism is given in section 4 and an overall discussion is given in section 5.

Together with the results in paper I, we have demonstrated that the QNMs of such open systems are both complete and orthogonal. Compared with conservative systems described by Hermitian operators in the usual sense, the only missing element is the lack of positivity, e.g. in the generalized norm. So apart from techniques such as the variational method, most

of the framework of mathematical physics based on eigenfunction expansion for *conservative* systems has been successfully generalized to a broad class of *nonconservative* systems in which the loss is due to leakage.

2. Linear space structure

2.1. Space of functions

In order to carry over the familiar mathematical formalism, one needs, in addition to the property of completeness derived in paper I, also the notion of orthogonality, and consequently a well-defined linear space structure. This is the principal task of the present paper, and we begin by specifying the function space and its elements.

Consider the set, Γ, of function pairs $(\phi, \hat{\phi})$ defined in paper I, where each of ϕ and $\hat{\phi}$ is defined on $[0, a]$, ϕ and $\hat{\phi}/\rho$ are differentiable, and the two functions satisfy conditions (1.9)–(1.11) in paper I. The set, Γ, forms a linear space under addition and multiplication by complex scalars. We use a ket vector to denote the column vector

$$|\phi\rangle = \begin{pmatrix} \phi(x) \\ \hat{\phi}(x) \end{pmatrix}. \tag{2.1}$$

An eigenfunction or QNM is a function pair $(f, \hat{f}) \in \Gamma$ whose first component satisfies (1.2), and whose second component is given by (1.3). The complex value ω_n in (1.2) and (1.3) is the eigenvalue, with Im $\omega_n < 0$ describing the rate of decay of the amplitude. (We shall give an alternate characterization of a QNM in section 2.5.)

2.2. Inner product

Given two elements $(\phi, \hat{\phi})$ and $(\psi, \hat{\psi})$ of Γ, we define the generalized inner product by

$$\langle \psi | \phi \rangle = \mathrm{i}\left[\int_0^{a^+} \mathrm{d}x \, (\psi\hat{\phi} + \hat{\psi}\phi) + \psi(a)\phi(a) \right] \tag{2.2}$$

which is symmetric and linear in both the bra and ket vectors (rather than conjugate linear in the bra vector). The inner product should not be regarded as a matrix product between the row vector representing the bra and the column vector representing the ket; it may be better to write the inner product as (ψ, ϕ), and matrix elements as $(\psi, \mathcal{H}\phi)$ etc. However, with this caveat on the notation, we choose to retain the bra–ket rotation for the more apparent parallel with quantum mechanics. Using (1.3), it is readily seen that the inner product of a QNM with itself agrees exactly with the generalized norm (1.5); the advantage of (2.2) is that it makes no reference to any eigenvalues; this is possible only because the second component appears. Moreover, the projection formula (1.6) for the eigenfunction expansion can now be written as

$$a_n = \frac{\langle f_n | \phi \rangle}{\langle f_n | f_n \rangle} = \frac{1}{2\omega_n} \langle \boldsymbol{f}_n | \phi \rangle. \tag{2.3}$$

2.3. Operators on the linear space

A linear operator is valid only if it maps Γ into Γ, i.e. it maps into function pairs $(\psi, \hat{\psi})$ that satisfy conditions (1.9)–(1.11) in paper I. Consider for example a 'potential' operator $\mathcal{V}$, where $|\psi\rangle = \mathcal{V}|\phi\rangle$ is defined by

$$\begin{pmatrix} \psi(x) \\ \hat{\psi}(x) \end{pmatrix} = \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix} \begin{pmatrix} \phi(x) \\ \hat{\phi}(x) \end{pmatrix}. \tag{2.4}$$

For (2.4) to be in Γ, one needs $\hat{\psi}(a^+) = -\psi'(a^+)$ (in addition to obvious conditions on differentiability for $x < a$ which we shall omit). This implies the following condition at a^+:

$$(V_{21} + V'_{11})\phi + (-V_{11} + V_{22} + V'_{12})\hat{\phi} + V_{12}\hat{\phi}' = 0 \tag{2.5}$$

where, of the four variables $\phi, \phi', \hat{\phi}, \hat{\phi}'$ we have eliminated ϕ' by using the condition that $(\phi, \hat{\phi})$ is in Γ; the remaining three variables are independent, leading to three conditions on $\mathcal{V}$ at $x = a^+$: $V_{21} + V'_{11} = 0$, $-V_{11} + V_{22} + V'_{12} = 0$, $V_{12} = 0$.

It is readily seen that the following are valid operators: the identity operator $\mathcal{I}$; the operator $\rho(x)\mathcal{I}$, using the condition that $\rho(a^+) = 1$; and any 'potential' operator $\mathcal{V}$ which vanishes outside a. The constant operator, for example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not valid.

Of particular importance is the time-dependent evolution, which can be written as

$$\frac{\partial}{\partial t}|\phi\rangle = -\mathrm{i}\mathcal{H}|\phi\rangle \tag{2.6}$$

where

$$\mathcal{H} = \mathrm{i}\begin{pmatrix} 0 & \rho(x)^{-1} \\ \partial_x^2 & 0 \end{pmatrix}. \tag{2.7}$$

The first component of (2.6) reproduces the identification of $\hat{\phi}$ as $\rho(x)\partial_t\phi$. If $|\psi\rangle = \mathcal{H}|\phi\rangle$, then we have $\psi(x) = \mathrm{i}\rho(x)^{-1}\hat{\phi}(x)$, $\hat{\psi}(x) = \mathrm{i}\partial_x^2\phi(x)$. For $\mathcal{H}$ to be a valid operator, we need $\hat{\psi} = -\psi'$ at a^+; it is readily verified that this indeed holds provided $\rho'(a^+) = 0$. Thus, time evolution keeps the wavefunction in Γ; this is physically obvious—an outgoing wavefunction evolves into an outgoing wavefunction.

The set of valid operators forms an algebra under addition, multiplication by complex scalars, and composition.

2.4. Self-adjoint operators

Given the generalized inner product, we define the adjoint $\mathcal{A}^+$ of any operator $\mathcal{A}$ as follows

$$\langle\psi|\{\mathcal{A}^+|\phi\rangle\} = \langle\phi|\{\mathcal{A}|\psi\rangle\} \tag{2.8}$$

and for a self-adjoint operator ($\mathcal{A}^+ = \mathcal{A}$), we adopt the following notation which is suggestive of left–right symmetry

$$\langle\psi|\{\mathcal{A}|\phi\rangle\} = \langle\phi|\{\mathcal{A}|\psi\rangle\} \equiv \langle\psi|\mathcal{A}|\phi\rangle. \tag{2.9}$$

The adjoint operator is defined without complex conjugation, so if $\mathcal{A}$ is self-adjoint, then so is $\alpha\mathcal{A}$ for any complex number α.

Strictly speaking $\langle\psi|\phi\rangle$ should be called a bilinear map rather than an inner product, and the matrix $\mathcal{A}$ is symmetric rather than self-adjoint. Nevertheless we adopt the slightly loose nomenclature to emphasize the parallel with quantum mechanics.

The time-evolution operator $\mathcal{H}$ is self-adjoint. To see this, condition (2.9) becomes, in this case,

$$\int_0^{a^+} \mathrm{d}x\,[\psi\partial_x^2\phi + \hat{\psi}\rho^{-1}\hat{\phi}] + \psi(a)\hat{\phi}(a) = \int_0^{a^+} \mathrm{d}x[\phi\partial_x^2\psi + \hat{\phi}\rho^{-1}\hat{\psi}] + \phi(a)\hat{\psi}(a) \tag{2.10}$$

where we have already used $\rho(a^+) = 1$ to simplify the surface terms. This condition is indeed satisfied, since the surface terms present exactly allow integration by parts. Usually (say in quantum mechanics), the hermiticity of the Hamiltonian operator is intimately related to the conservation of probability; here we have succeeded in casting the dynamics of a nonconservative system in terms of a self-adjoint evolution operator. The self-adjoint property is key to the rest of the development.

2.5. Eigenfunctions

The eigenfunctions, or QNMs, can now be defined simply by $\mathcal{H}|\boldsymbol{f}_n\rangle = \omega_n|\boldsymbol{f}_n\rangle$, which incorporates both the differential equation for f_n as well as the definition of $\hat{f}_n$. Since $\mathcal{H}$ is self-adjoint under the definition introduced, it is easily shown, by the usual procedure, that if $\omega_m \neq \omega_n$, then $\langle \boldsymbol{f}_m|\boldsymbol{f}_n\rangle = 0$. This leads immediately to the uniqueness of the completeness sum (1.1). Incidentally, the expansions labelled as methods B and C in paper I are not unique expansions for the function ϕ.

It is seen that the mathematical structure is in place to carry over essentially all the familiar tools based on eigenfunction expansions. The only exception is the lack of a positive-definite norm, and with it a simple probability or energy interpretation. This is hardly surprising since on the interval I (as in any finite parts of space), probability or energy is not conserved. We should also mention that the completeness property from paper I motivates but is not necessary for the definition of self-adjointness, or for the consequent properties.

3. Time-independent perturbation

3.1. Analogy with quantum mechanics

The mathematical formalism lends itself immediately to the development of a variety of mathematical tools. Here we only show the case of the time-independent perturbation theory. Let $\rho(x)^{-1}$ be changed from the original value $\rho_0(x)^{-1}$ to $\rho(x)^{-1} = \rho_0(x)^{-1}[1 + \mu V(x)]$, in which μ is a formal small parameter. The perturbation $V(x)$ is assumed to vanish outside the interval $I = [0, a]$. The problem is to find the eigenvalues and eigenfunctions of the system as a power series in μ, assuming that the original unperturbed problem described by $\rho_0(x)$ has already been solved. A physical example could be a laser cavity with output coupling which has been subject to a density perturbation [2].

The eigenvalue problem can be stated in terms of $\mathcal{H} = \mathcal{H}_0 + \mu\mathcal{V}$, where

$$\mathcal{H}_0 = \mathrm{i}\begin{pmatrix} 0 & \rho_0(x)^{-1} \\ \partial_x^2 & 0 \end{pmatrix} \tag{3.1}$$

$$\mathcal{V} = \mathrm{i}\begin{pmatrix} 0 & \rho_0(x)^{-1}V(x) \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

We assume that the unperturbed problem $\mathcal{H}_0|\boldsymbol{f}_n^{(0)}\rangle = \omega_n^{(0)}|\boldsymbol{f}_n^{(0)}\rangle$ has been solved, and express the exact quantities in a power series in μ:

$$\omega_n = \omega_n^{(0)} + \mu\omega_n^{(1)} + \cdots \tag{3.3}$$

$$|\boldsymbol{f}_n\rangle = C_n \sum_m |\boldsymbol{f}_m^{(0)}\rangle a_{mn} \tag{3.4}$$

$$a_{mn} = a_{mn}^{(0)} + \mu a_{mn}^{(1)} + \cdots. \tag{3.5}$$

In these equations, the superscript denotes the order in μ, and $a_{mn}^{(0)} = \delta_{mn}$. The factor C_n is inserted to allow a choice of normalization of the exact eigenstates $|\boldsymbol{f}_n\rangle$.

The problem is exactly parallel to the analogous problem $H|\psi\rangle = E|\psi\rangle$ in quantum mechanics, where $H = H_0 + \mu V$. First write the familiar expressions in a form independent of normalization, by dividing by $\langle n|n\rangle$ etc. Notice now that these factors are $\langle n|n\rangle = 2\omega_n^{(0)}$. Matrix elements become $\langle n|V|m\rangle \to \langle \boldsymbol{f}_n^{(0)}|\mathcal{V}|\boldsymbol{f}_m^{(0)}\rangle$, where the latter is evaluated using

$$\mathcal{V}|\boldsymbol{f}_m^{(0)}\rangle = \begin{pmatrix} \mathrm{i}\rho_0(x)^{-1}V(x)\hat{f}_m^{(0)}(x) \\ 0 \end{pmatrix}. \tag{3.6}$$

Using the definition of the inner product in (2.3), and expressing $\hat{f}_m^{(0)}(x)$ in terms of $f_m^{(0)}(x)$, we then obtain

$$\begin{aligned} \langle \boldsymbol{f}_n^{(0)}|\mathcal{V}|\boldsymbol{f}_m^{(0)}\rangle &= \omega_n^{(0)}\omega_m^{(0)}V_{nm} \\ &\equiv \omega_n^{(0)}\omega_m^{(0)}\int_0^\infty \mathrm{d}x\, f_n^{(0)}(x)\rho_0(x)V(x)f_m^{(0)}(x). \end{aligned} \tag{3.7}$$

With these replacements and some trivial arithmetic, the familiar results for quantum mechanics then map onto the following for the problem at hand.

$$\omega_n^{(1)} = \tfrac{1}{2}\omega_n^{(0)}V_{nn} \tag{3.8}$$

$$\omega_n^{(2)} = \frac{1}{4}\sum_{m\neq n} V_{nm}\frac{\omega_n^{(0)}\omega_m^{(0)}}{\omega_n^{(0)} - \omega_m^{(0)}}V_{mn} \tag{3.9}$$

etc. while for the eigenfunctions we have

$$a_{mn}^{(1)} = \frac{\omega_n^{(0)}}{2(\omega_n^{(0)} - \omega_m^{(0)})}V_{nm} \qquad m \neq n \tag{3.10}$$

etc. The transcription is readily written down for higher orders. The first-order result is valid even in circumstances where the QNMs are not complete.

If we want the exact state to be normalized to $\langle \boldsymbol{f}_n|\boldsymbol{f}_n\rangle = 2\omega_n$ in analogy to $\langle \boldsymbol{f}_n^{(0)}|\boldsymbol{f}_n^{(0)}\rangle \equiv \langle n|n\rangle = 2\omega_n^{(0)}$, then the normalization constant C_n in (3.6) should be chosen as $C_n = 1 + (\frac{1}{4})\mu V_{nn} + \cdots$.

The first-order correction has been known for a long time in the context of the Schrödinger equation [3], though the generalized norm that appears was stated in a way that required a process of regularization rather than a compensation by a surface term, the latter being much more computationally direct. The first-order formalism has been generalized to the electromagnetic case, including the presence of degeneracies [4]. The formalism to higher orders has also been presented previously [5–7]. The earlier work did not draw on such a close parallel with the familiar formalism, and as a result the derivation was more complicated. Moreover, the perturbation was phrased as a correction to $\rho(x)$, rather than as a correction to $\rho(x)^{-1}$, the latter now being seen as much more natural in view of the analogy with quantum mechanics presented in this section. By straightforward computation, the earlier results [5–7] can be shown to be equivalent to those presented here; the demonstration requires the sum rules (2.3) and (2.4) in paper I.

Despite the formal similarity with quantum mechanics, the generalization is nontrivial, because the present perturbation formulae give the corrections to both Re ω and Im ω, i.e. to both the resonance position and the width; the latter contains interesting features that have no counterpart in the familiar conservative case.

Examples of the application of the perturbation theory have been given elsewhere [4–10] and include comparisons with exactly soluble models, with brute force numerical calculations, and also with experiments. A further example is given in the next section.

3.2. Improvement of the convergence rate

While the above results are entirely correct, they can be cast in a slightly different form with better convergence in the sum over intermediate states. For this purpose we note that identities (2.3) and (2.4) in paper I lead to

$$\sum_m V_{am}\frac{1}{\omega_m}V_{mb} = 0 \tag{3.11}$$

$$\tfrac{1}{2}\sum_m V_{am}V_{mb} = (V^2)_{ab} \tag{3.12}$$

in which the matrix elements of V^2 are defined in the same manner as (3.7). The first of these has no counterpart in conservative systems; it originates from the fact that these matrix elements are only sensitive to the first component, which by themselves (i.e. without the second component) do not have unique expansions. The second identity is formally the same as in the conservative case, except for the factor of $\frac{1}{2}$, which is related to the doubling of eigenfunctions—in the limit of zero leakage, the states n and $-n$ are the same.

Consider, for example, the sum in (3.9) and write

$$\frac{\omega_m^{(0)}}{\omega_n^{(0)}-\omega_m^{(0)}} = -1 - \frac{\omega_n^{(0)}}{\omega_m^{(0)}} + \frac{\omega_n^{(0)2}}{\omega_m^{(0)}(\omega_n^{(0)}-\omega_m^{(0)})}. \tag{3.13}$$

Inserting into (3.9), we find that $\omega_n^{(2)} = (\frac{1}{4})(A+B+C)$, where

$$\begin{aligned} A &= -\omega_n^{(0)}\sum_{m\neq n} V_{nm}V_{mn} \\ &= -\omega_n^{(0)}[2(V^2)_{nn} - (V_{nn})^2] \end{aligned} \tag{3.14}$$

in which we have used the sum rule (3.12), and

$$\begin{aligned} B &= -\omega_n^{(0)2}\sum_{m\neq n} V_{nm}\frac{1}{\omega_m^{(0)}}V_{mn} \\ &= \omega_n^{(0)}(V_{nn})^2 \end{aligned} \tag{3.15}$$

in which we have used the sum rule (3.11). Finally,

$$C = \omega_n^{(0)3}\sum_{m\neq n} V_{nm}\frac{1}{\omega_m^{(0)}(\omega_n^{(0)}-\omega_m^{(0)})}V_{mn}. \tag{3.16}$$

The remaining sum C is now improved by two powers of $\omega_m^{(0)}$ compared with (3.9). This technique can be applied to the eigenvalue to any order.

The improved rate of convergence comes about through identities in the sum over states. For the eigenfunction, for example

$$|\boldsymbol{f}_n^{(1)}\rangle = C_n\sum_m |\boldsymbol{f}_m^{(0)}\rangle a_{mn}^{(1)} \tag{3.17}$$

the expansion is unique, and there is no way to improve the convergence. However, for all practical purposes, we only need the first component, for which

$$\begin{aligned} f_n^{(1)}(x) &= C_n\sum_{m\neq n} f_m^{(0)}(x)a_{mn}^{(1)} \\ &= \frac{C_n}{2}\int_0^{a^+}\mathrm{d}y\left\{\sum_{m\neq n} f_m^{(0)}(x)\frac{\omega_m^{(0)}}{\omega_n^{(0)}-\omega_m^{(0)}}f_m(y)\right\}\rho_0(y)V(y)f_n^{(0)}(y). \end{aligned} \tag{3.18}$$

Again using the sum rule (2.3) in paper I, and noticing in this case that terms proportional to $f_n^{(0)}$ can be absorbed into a redefinition of the normalization constant C_n, we find that, as far as the first component is concerned, $a_{mn}^{(1)}$ can be replaced by

$$a_{mn}^{(1)} = \frac{\omega_n^{(0)^2}}{2\omega_m^{(0)}(\omega_n^{(0)} - \omega_m^{(0)})} V_{nm} \tag{3.19}$$

which again shows an improvement in the sum over intermediate states. The same trick can be used to any order.

4. Example

To illustrate the time-independent perturbation theory developed in the last section, we consider a simple example. We take the unperturbed system to be the dielectric rod in paper I, and let $\rho(x)$ have a further barrier of height $\sim \mu$ in the region $0 < b < x < a$. Specifically

$$\frac{1}{\rho(x)} = \begin{cases} n_0^{-2} & \text{if } 0 \leqslant x < b < a \\ n_0^{-2} - \mu & \text{if } b < x < a \\ 1 & \text{if } a < x \end{cases} \tag{4.1}$$

where $\rho_0(x)^{-1}$ corresponds to the case $\mu = 0$. The eigenfunctions and eigenvalues of this system can be solved analytically for any μ. Figure 1 shows their positions in the ω-plane for $\mu = 0.093$, compared with the case $\mu = 0$. The two are qualitatively quite different, in that for the former case Im ω_n are oscillatory in the mode number n.

The first-order and second-order perturbative results are also shown in figure 1. It is seen that the qualitative behaviour is captured by the perturbation series. More importantly, figure 2 shows, for one state, the absolute value of the remaining error between the exact and the second-order results, as a function of μ. The remaining error for the first-order result is seen to scale as μ^2, and for the second-order result as μ^3, as they should. The errors are small for moderate values of μ. Similar numerical verifications have also been carried out in a few cases to higher orders.

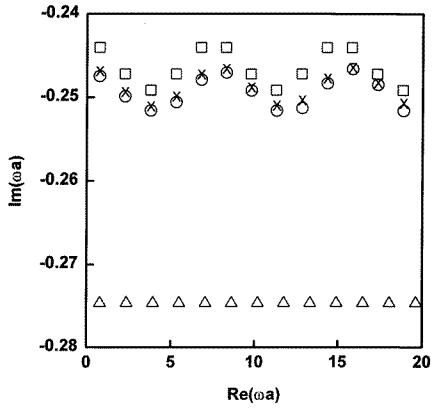

Figure 1. Positions of the unperturbed (triangles), first-order (squares), second-order (circles) and exact (crosses) eigenvalues for the example in (4.1), when $n_0 = 2$, $a = 1$, $b = 0.8$, $\mu = 0.093$.

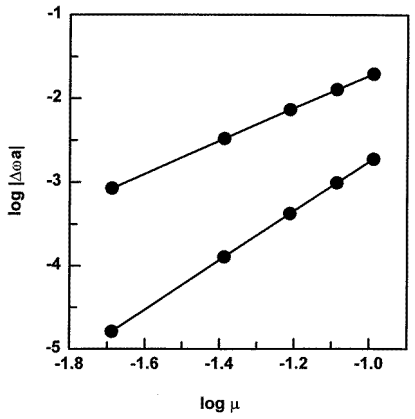

Figure 2. The absolute value of the remaining error in the eigenvalue versus μ. The upper curve is the first-order result. The lower curve is the second-order result. The fifth QNM with Re $(\omega a) > 0$ is chosen. The parameters are $n_0 = 2$, $a = 1$, $b = 0.8$.

We may stress that the generalized inner product (which is not conjugate linear in the bra vector) is responsible for the appearance of nontrivial phases, which are in turn essential in order to account for the imaginary parts of the eigenvalues.

5. Discussion

The class of problems considered in this and the previous paper [1] concern wavemechanics in a system S (the interval $I = [0, a]$) which is coupled to a bath B (the rest of the real line (a, ∞)). As a result, S by itself is dissipative. It is therefore useful to place the discussion in perspective by comparing it with treatments of other dissipative systems.

The proper treatment of dissipative systems in the quantum domain is, by now, well developed [11]. One must consider $S+B$ from the start; the two together form a conservative system to which the standard tools apply. The bath degrees of freedom are then eliminated either by solving the equations of motion or by integrating them out in a path integral or equivalent formalism [11], leaving only the few degrees of freedom of S, and their dynamics contains dissipation in a manner that is guaranteed to be consistent with basic principles.

The results in this paper and in paper I accomplish analogous tasks for the wave equation. Reference to the bath coordinates, i.e. $\phi(x), a < x < \infty$ is removed, and the remaining degrees of freedom, i.e. $\phi(x), 0 \leqslant x \leqslant a$ are 'few' in number (namely, discrete) and are described by dynamics that contains dissipation. There are, however, two essential aspects in which the class of problems considered here differ from the more familiar models [11], and which makes it impossible to map those results directly to the problems at hand. First, the system and the bath are now coupled by boundary conditions (e.g. $\phi(a^-) = \phi(a^+)$), and such a coupling cannot be switched off. Secondly, in the limit of zero leakage (e.g. achieved by clamping the point $x = a$), one loses one degree of freedom, namely $\phi(a)$. Therefore a re-examination of the theory is needed, in which the removal of the 'outside' or bath coordinates is carried out from first principles. This has been achieved in both paper I and this one. In particular, under suitable circumstances it has been shown that the system can be described completely, and in close analogy with conservative systems, by the spectrum of QNMs. The property of dissipation is contained in these discrete QNMs themselves, especially in Im ω_n. The remarkable property is that apart from this feature,

almost nothing needs to be changed, and the familiar formalism based on eigenfunction expansions for Hermitian systems can be carried over. These results are highly nontrivial, in that if either the discontinuity condition or the 'no tail' condition is violated, then the QNMs would not be complete.

The formalism opens the way to many applications, which will be considered elsewhere. The possible generalization to multi-channels, higher dimensions, second quantization and other second-order hyperbolic systems will also be considered separately.

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